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INTEGRATION BY PARTS FORMULA INVOLVING MALLIAVIN DERIVATIVES AND SOLUTIONS TO DELAY SDE'S

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ABSTRACT

In the present work we have established an integration by parts formula of higher order Malliavin derivatives of solutions to delay stochastic differential equations. In a squeal work we will use this integration by parts formula in some applications concerning densities of distributions of solutions of delay (as well as ordinary) stochastic differential equations with possibly discontinuous initial data. Also, this integration by parts formula can be used to extended the formulas in the work by Bally and Talay to include delay as well as ordinary SDE's.

KEYWORDS: Stochastic Differential Equations, Malliavin Calculus, Euler Scheme for delay SDE's, Integration by Parts, Densities of Distributions.

INTRODUCTION

In Chapter 1 of the Ph.D. thesis of Ahmed (15) we have proved the existence and uniqueness of a solution for certain types of delay (functional) stochastic differential equations (delay SDE's) with discontinuous initial data,see also(1),(9) and the web cite www.sfde.math.siu.edu. See the delay SDE (1.1) in the present work. Here we establish an integration by parts formula involving solutions to such type of delay (functional) SDE's. The integration by parts formula which we establish can be used to extend the formulas in (2) and(3) to include delay SDE's as well as ordinary SDE's. In this work we also establish some other useful applications to delay SDE's. Generally speaking we can say that our work extends the first three chapters of the work by Norris to include delay SDE's as well as ordinary SDE's; see Theorems 2.3, 3.1 and 3.2 in(10) . We will also show in a sequal paper to this work that the distribution of the solution process has smooth density. Also we will establish an integration by parts formula involving Malliavin derivatives of higher order.

NOTATIONS AND DEFINITIONS

The following notations and definitions will be used throughout this work: $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space; T is a positive real number; $\{\mathcal{F}_t\}_{t\in[0,T]}$ is an increasing family of sub-σ algebras of $\mathcal{F}$, each of which contains all null subsets of Ω; $\mathbb{N}$ is the set of natural numbers; $W=(W^1,\dots,W^r):[0,T]\times\Omega\to\mathbb{R}^r$ is a r-dimensional normalized Brownian motion. If X is a topological space, then $\mathcal{B}(X)$ denotes its Borel field. The symbol λ refers to the Lebesgue measure on $\mathbb{R}^d$, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, $d\in\mathbb{N}$.

Let G be a Banach space and let $\mathcal{A}$ be a sub-σ algebra of $\mathcal{F}$ containing all subsets of measure zero in $\mathcal{F}$, then $\mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P};G)$ denotes the space of all functions $f:\Omega\to G$ which are $\mathcal{A}$-$\mathcal{B}(G)$ measurable and are such that $\int_\Omega \|f\|_G^2\,d\mathbb{P}<\infty$.

The symbol $L^2(\Omega,\mathcal{A},\mathbb{P};G)$ denotes the Banach space (with norm determined by $\|f\|_{L^2}^2=\int_\Omega \|f(\omega)\|_G^2\,d\mathbb{P}$) of all equivalence classes of functions $f:\Omega\to G$ which are $\mathcal{A}$-$\mathcal{B}(G)$ measurable and which are such that $\int_\Omega \|f\|_G^2\,d\mathbb{P}<\infty$. The symbol $L(\mathbb{R}^m,\mathbb{R}^n)$ ($m,n\in\mathbb{N}$) denotes the space of all linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. The symbol J refers to the interval $[-1,0)$, and $\mathcal{H}(J)$ or $\mathcal{B}(J)$ refers to the Borel field on J.

If $X:[-1,T]\times\Omega\to\mathbb{R}^d$ is a process, then for each $t\in[0,T]$ and $\omega\in\Omega$ we define the map: $X_t:\Omega\to\mathcal{L}^2(J,\mathbb{R}^d)$ by $X_t(\omega)(s)=X(t+s,\omega)$ for all $s\in J$ and almost all ω. For each $0\le t\le T$ we write $\|(X(t),X_t)\|^2=\|X(t)\|^2+\|X_t\|^2$. Let the function V belong to $\mathcal{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$, θ belong to $\mathcal{L}^2(J\times\Omega,\mathcal{H}(J)\otimes\mathcal{F}_0,\lambda\otimes\mathbb{P};\mathbb{R}^d)$, and for $\ell=1,2,\dots,r$ let f ,g^ℓ be functions from $[0,T]\times\Omega\times\mathbb{R}^d\times\mathcal{L}^2(J,\mathbb{R}^d)$ to $\mathbb{R}^d$. Then a process $X:[-1,T]\times\Omega\to\mathbb{R}^d$ is called a solution of the delay SDE with integral form

$$X(t)=\begin{cases} V+\int_0^t f(u,X(u),X_u)du+\sum_{\ell=1}^r\int_0^t g^\ell(u,X(u),X_u)\,dW^\ell(u), & 0\le t\le T,\\ \theta(t), & t\in J,\end{cases} \tag{1.1}$$

if

(i) X is $\mathcal{B}([0,T])\otimes\mathcal{F}$-$\mathcal{B}(\mathbb{R}^d)$ measurable;

(ii) For each $t\in[0,T]$, the process $X(t,\cdot)$ is $\mathcal{F}_t$-$\mathcal{B}(\mathbb{R}^d)$ measurable, and for each $t\in J$, the process $X(t,\cdot)$ is $\mathcal{F}_0$-$\mathcal{B}(\mathbb{R}^d)$ measurable;

(iii) $X\in\mathcal{L}^2([-1,T]\times\Omega,\mathcal{H}\times\mathcal{F},\lambda\times\mathbb{P};\mathbb{R}^d)$,

(iv) X satisfies the delay SDE ([1.1.1]).

The following conditions are sufficient for the existence of a unique solution to (1.1) (see [1] and [15]).

(i) $V\in\mathcal{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$.

(ii) $\theta\in\mathcal{L}^2(J\times\Omega,\mathcal{H}\otimes\mathcal{F}_0,\lambda\otimes\mathbb{P},\mathbb{R}^d)$.

(iii) $f,g^\ell:[0,T]\times\Omega\times\mathbb{R}^d\times\mathcal{L}^2(J,\mathbb{R}^d)\to\mathbb{R}^d$ are such that

(a) f and g^ℓ are $\mathcal{B}([0,T])\otimes\mathcal{F}\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}\big(L^2(J,\mathbb{R}^d)\big)$-$\mathcal{B}(\mathbb{R}^d)$ measurable.

(b) For each $t\in[0,T]$, the stochastic variables $f(t,\cdot,\cdot,\cdot)$ and $g^\ell(t,\cdot,\cdot,\cdot)$ are $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{B}\big(\mathcal{L}^2(J,\mathbb{R}^d)\big)$-$\mathcal{B}(\mathbb{R}^d)$ measurable.

(c)There exists a constant K and a function $\zeta\in\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ such that $|f(t,\omega,s,h)|+\sum_{\ell=1}^r\big|g^\ell(t,\omega,s,h)\big|\le K(|s|+\|h\|+|\zeta(\omega)|)$ (1.2) (1.2)

for almost all ω and for all $t\in[0,T]$; $s\in\mathbb{R}^d$ and h belongs to $\mathcal{L}^2(J,\mathbb{R}^d)$.

(d) There exists a constant K' such that, for almost all ω,

$$|f(t,\omega,s,h_1) - f(t,\omega,u,h_2)| + \sum_{\ell=1}^{r}\left|g^{\ell}(t,\omega,s,h_1) - g^{\ell}(t,\omega,u,h_2)\right| \leq K'(|s-u| + \|h_1 - h_2\|) \quad (1.3)$$

for all $t \in [0,T]$; for all $s, u \in \mathbb{R}^d$, and for all $h_1, h_2 \in \mathcal{L}^2(J,\mathbb{R}^d)$.

INTEGRATION BY PARTS FORMULA

In the beginning of this section we recall the following eight basic numbered equations and definitions, See(16) and(17) . For $(X(0), X_0) = (x,\xi) \in \mathbb{R}^d \times L^2(J,\mathbb{R}^d)$, let $v \mapsto D^v X^{x,\xi}(t)$, be the Malliavin derivative of the solution process $X^{x,\xi}(t)$. We write $D^v X_t^{x,\xi}(\vartheta) = D^v X^{x,\xi}(t+\vartheta)$ ($t \in [0,T], \vartheta \in J = [-1,0)$) for its time delay. In the following definition we give a precise definition of the Malliavin derivative of a real-valued functional F of Brownian motion.

I.Definition: Let $F((W(s))_{0\leq s\leq T})$ be a functional of r-dimensional Brownian motion, and let $v(t) = (v^1(t), \ldots, v^r(t))^* = \begin{pmatrix} v^1(t) \\ \vdots \\ v^r(t) \end{pmatrix}$ be a deterministic vector-valued function in $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$.

Then $D^v F((W(s))_{0\leq s\leq T})$ is given by the limit:

$$D^v F((W(s))_{0\leq s\leq T}) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(F\left(\left(W(s) + \varepsilon \int_0^s v(\sigma) d\sigma\right)_{0\leq s\leq s}\right) - F((W(s))_{0\leq s\leq t})\right). \quad (2.1)$$

The mapping $v \mapsto D^v F((W(s))_{0\leq s\leq T})$ is a linear map (functional) from the space $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$to $\mathbb{R}$. Here $\mathbb{R}^r \otimes \mathbb{R}^d$ denotes the space of all $r \times d$-matrices (r rows, d columns).

Notice that, for $v(t) = (v^1(t), \ldots, v^r(t))^\tau = \begin{pmatrix} v^1(t) \\ \vdots \\ v^r(t) \end{pmatrix}$ be a deterministic matrix-valued function in $L^2([0,T], \mathbb{R}^r \otimes \mathbb{R}^d)$, $U^v(t)$ can be considered as a $d \times d$-matrix where each entry is an $\mathbb{R}$-valued adapted stochastic process; U_t^v can be considered as a $d \times d$-matrix where each entry is an $L^2(J,\mathbb{R})$-valued adapted stochastic process. If $M = (m_{jk})_{1\leq j\leq d,\, 1\leq k\leq r}$ is a real $d \times r$ matrix, then $M^\tau = (m_{kj})_{1\leq k\leq r,\, 1\leq j\leq d}$ denotes its transposed: it is $r \times d$ matrix with entries m_{kj}.

The process $D^v X_t^{x,\xi}(\cdot)$ satisfies the following delay stochastic differential equation:

$$\begin{aligned} dD^vX_t(\vartheta) &= dD^vX(t+\vartheta) \\ &= \left(\frac{\partial f}{\partial x}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})D^vX(t+\vartheta) \right. \\ &\quad \left. + \int_J \frac{\partial f}{\partial \xi}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})(\varphi)D^vX_{t+\vartheta}(\varphi)\, d\varphi\right) dt \\ &\quad + \sum_{\ell=1}^r \frac{\partial g^\ell}{\partial x}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})D^vX(t+\vartheta)dW^\ell(t+\vartheta) \\ &\quad + \sum_{\ell=1}^r \int_J \frac{\partial g^\ell}{\partial \xi}(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})(\varphi)D^vX_{t+\vartheta}(\varphi)\, d\varphi\, dW^\ell(t+\vartheta) \\ &\quad + \sum_{\ell=1}^r g^\ell(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})v^\ell(t+\vartheta, X(t+\vartheta), X_{t+\vartheta})dt, \end{aligned} \tag{2.2}$$

whereϑ belongs to J. If $t+\vartheta$ belongs to J we replace $t+\vartheta$ with 0 in(2.2) ([E: SDEDP]). If $\vartheta = 0$ we obtain the delay stochastic differential equation for the process $D^vX(t)$:

$$\begin{aligned} &dD^vX(t) \\ &= \left(\frac{\partial f}{\partial x}(t, X(t), X_t)D^vX(t) + \int_J \frac{\partial f}{\partial \xi}(t, X(t), X_t)(\vartheta)D^vX_t(\vartheta)d\vartheta\right) dt \\ &+ \sum_{\ell=1}^r \left(\frac{\partial g^\ell}{\partial x}(t, X(t), X_t)D^vX(t) + \int_J \frac{\partial g^\ell}{\partial \xi}(t, X(t), X_t)(\vartheta)D^vX_t(\vartheta)d\vartheta\right) dW^\ell(t) \\ &+ \sum_{\ell=1}^r g^\ell(t, X(t), X_t)v^\ell(t, X(t), X_t)dt. \end{aligned} \tag{2.3}$$

We also write $U_{11}^{x,\xi}(t) = \frac{\partial}{\partial x}X^{x,\xi}(t)$, and $U_{12}^{x,\xi}(t) = \frac{\partial}{\partial \xi}X^{x,\xi}(t)$. In addition, we write $U_{21}^{x,\xi}(t) = \frac{\partial}{\partial x}X_t^{x,\xi} = U_{11,t}^{x,\xi}$ (the delay of $U_{11}^{x,\xi}(t)$), and $U_{22}^{x,\xi}(t) = \frac{\partial}{\partial \xi}X_t^{x,\xi} = U_{12,t}^{x,\xi}$, the delay of the process $U_{12}^{x,\xi}(t)$. The matrix $U_{11}^{x,\xi}(t)$ can be identified with an operator from $\mathbb{R}^d$ to itself, the matrix $U_{12}^{x,\xi}(t)$ can be considered as an linear mapping from $L^2(J,\mathbb{R}^d)$ to $\mathbb{R}^d$, the matrix $U_{21}^{x,\xi}(t)$ as a mapping from $\mathbb{R}^d$ to $L^2(J,\mathbb{R}^d)$, and, finally, $U_{22}^{x,\xi}(t)$ as a mapping from $L^2(J,\mathbb{R}^d)$ to itself. Notice that $U_{11}^{x,\xi}(t)$ can be considered as $d \times d$-matrix where each entry is an $\mathbb{R}$-valued adapted stochastic process; $U_{12}^{x,\xi}(t)$ can be considered as $d \times d$-matrix where each entry is an $L^2(J,\mathbb{R})$-valued adapted stochastic process. To be precise, write the solution process as a d-vector $X^{x,\xi}(t) = \left(X_1^{x,\xi}(t), \dots, X_d^{x,\xi}(t)\right)$, and consider the mapping ($1 \le j, k \le d$)

$$\xi_k \to X_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k,\xi_{k+1},\dots,\xi_d)}(t), \tag{2.4}$$

which is a mapping from $L^2(J,\mathbb{R})$ to $\mathbb{R}$, and where each variable ξ_ℓ, $\ell \neq k$ is a fixed function in $L^2(J,\mathbb{R})$. The derivative of the function in (2.4) can be considered as a continuous linear functional on $L^2(J,\mathbb{R})$. Therefore it can be represented as an inner-product with a function in $L^2(J,\mathbb{R})$, which is denoted by $\frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}$.

Consequently, we write

$$\frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}(\eta) = \lim_{h\to 0}\frac{X_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k+h\eta,\xi_{k+1},\dots,\xi_d)}(t)-X_j^{x,(\xi_1,\dots,\xi_{k-1},\xi_k,\xi_{k+1},\dots,\xi_d)}(t)}{h}$$
$$= \int_J \eta(\varphi)\frac{\partial X_j^{x,\xi}(t)}{\partial \xi_k}(\varphi)d\varphi, \quad \eta\in L^2(J,\mathbb{R}). \tag{2.5}$$

After giving a brief introduction to our work, we are now ready to continue the work that we have started in . (16)

Here, and in the sequel, we write $f(t)$ and $g^{\ell}(t)$ instead of $f\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)$ and $g^{\ell}\left(t, X^{x,\xi}(t), X_t^{x,\xi}\right)$ respectively. For a concise formulation of the stochastic differential equation for the matrix-valued process $(U(t): t \geq)$ and its inverse we introduce the following *stochastic differentials*:

$$h_x(t) = \frac{\partial f}{\partial x}(t)dt + \sum_{\ell=1}^{r}\frac{\partial g^{\ell}}{\partial x}(t)dW^{\ell}(t);$$
$$h_\xi(t) = \frac{\partial f}{\partial \xi}(t)dt + \sum_{\ell=1}^{r}\frac{\partial g^{\ell}}{\partial \xi}(t)\,dW^{\ell}(t) \qquad (2.6)\ (2.7)\ (2.8)$$
$$h_\xi(t,\vartheta) = \frac{\partial f}{\partial \xi}(t,\vartheta)dt + \sum_{\ell=1}^{r}\frac{\partial g^{\ell}}{\partial \xi}(t,\vartheta)dW^{\ell}(t)$$

The following theorem is an extension of Theorem 2.3 (Integration by Parts Formula) of the work of Norris to include delay as well as ordinary SDE's.

2.Theorem:(Integration by Parts Formula) [T:1] Let $W(t)$ be r-dimensional Brownian motion and let $X(t)$ be the solution of

$$dX(t) = f(t,X(t),X_t)dt + \sum_{\ell=1}^{r} g^{\ell}(t,X(t),X_t)dW^{\ell}(t),$$

with $(X(0),X_0) = (x,\xi)$, where $f, g^1, g^2, \dots, g^r$ are maps $[0,T]\times\mathbb{R}^d\times L^2(J,\mathbb{R}^d)\to\mathbb{R}^d$ and $J=[-1,0)$ and T is a positive real number. Let $v^{\ell}:[0,T]\times\mathbb{R}^d\times L^2(J,\mathbb{R}^d)\to\mathbb{R}^d$, $(\ell = 1,2,\dots,r)$ be C^{∞}, with all derivatives of polynomial growth. Then the linear delay SDE

$$\begin{aligned} d\,D^vX(t) &= \frac{\partial}{\partial x}f(t,X(t),X_t)D^vX(t)\,dt + \int_J \frac{\partial}{\partial \xi}f(t,X(t),X_t)(\vartheta)D^vX_t(\vartheta)\,d\vartheta\,dt \\ &+ \sum_{\ell=1}^{r}\frac{\partial}{\partial x}g^{\ell}(t,X(t),X_t)D^vX(t)dW^{\ell}(t) \\ &+ \sum_{\ell=1}^{r}\int_J\frac{\partial}{\partial \xi}g^{\ell}(t,X(t),X_t)(\vartheta)D^vX_t(\vartheta)\,d\vartheta\,dW^{\ell}(t) \\ &+ \sum_{\ell=1}^{r} g^{\ell}(t,X(t),X_t)v^{\ell}(t,X(t),X_t)dt. \end{aligned} \tag{2.9}$$

with $(D^vX(0), D^vX_0) = 0 \in \left(\mathbb{R}^d\otimes\mathbb{R}^r\times L^2(J,\mathbb{R}^d\otimes\mathbb{R}^r)\right)$ has a unique solution with $\sup_{s\le t}|D^vX(s)|\in L^p(\Omega,\mathcal{F}_t,P)$, and with $\sup_{s\le t}|D^vX_s|\in L^p(\Omega\times J,\mathcal{F}_t\otimes\mathcal{B},P\times\lambda)$ for all $t\geq 0$ and for all $p<\infty$. for all $t\geq 0$ and for all $p<\infty$. We also write D^vX_t for the delay process $\vartheta\mapsto D^vX(t+\vartheta)$. Furthermore, for any function $\Phi: U_0\to\mathbb{R}$, where U_0 is an open subset of

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$[0,T]\times\mathbb{R}^d\times L^2(J,\mathbb{R}^d)$ with $(t,X(t),X_t)\in U_0$ a.s. such that Φ is differentiable, and $D\Phi(t,X(t),X_t)$ and $\Phi(t,X(t),X_t)$ belong to $L^2(\Omega,\mathcal{F}_t,\mathbb{P})$,

$$\mathbb{E}\left[\frac{\partial}{\partial x}\Phi(t,X(t),X_t)D^vX(t)+\int_J\frac{\partial}{\partial\xi}\Phi(t,X(t),X_t)(\vartheta)D^vX_t(\vartheta)d\vartheta\right] = \sum_{\ell=1}^r\mathbb{E}\left[\Phi(t,X(t),X_t)\int_0^t v^\ell(t,X(t),X_t)dW^\ell(t)\right], \tag{2.10}$$

where a choice for v^ℓ could be

$$v^\ell(s,X(s),X_s)=\left(V(s)g^\ell(s,X(s),X_s)\right)^\tau. \tag{2.11}$$

Specializing ([E: int.formula]) to the partial derivatives $\frac{\partial}{\partial x_k}$and $\frac{\partial}{\partial\xi_k}$, $1\le k\le d$, we obtain

$$\mathbb{E}\left[\frac{\partial}{\partial x_k}\Phi(t,X(t),X_t)D^vX(t)+\int_J\frac{\partial}{\partial\xi_k}\Phi(t,X(t),X_t)(\vartheta)D^vX_t(\vartheta)d\vartheta\right] = \sum_{\ell=1}^r\mathbb{E}\left[\Phi(t,X(t),X_t)\int_0^t v_k^\ell(t,X(t),X_t)dW^\ell(t)\right], \tag{2.12}$$

Proof. Suppose f, g^1, $g^2,\dots,g^d$ are maps from $[0,T]\times\mathbb{R}^d\times L^2(J,\mathbb{R}^d)$ to $\mathbb{R}^d$ satisfying the following Lipschitz and Linear growth conditions

$$|f(t,y_2,\eta_2)-f(t,y_1,\eta_1)|+\sum_{\ell=1}^r|g^\ell(t,y_2,\eta_2)-g^\ell(t,y_1,\eta_1)| \le K(|y_2-y_1|+\|\eta_2-\eta_1\|) \tag{2.13}$$

for all $(y_i,\eta_i)\in\mathbb{R}^d\times L^2(J,\mathbb{R}^d)$, $i=1,2$

$$|f(t,y,\eta)|+\sum_{\ell=1}^r|g^\ell(t,y,\eta)| \quad \le K(|y|+||\eta||+1) \tag{2.14}$$

for all $(y,\eta)\in\mathbb{R}^d\times L^2(J,\mathbb{R}^d)$. Then by the existence theorem in the delay SDE

$$dX(t)=f(t,X(t),X_t)dt+\sum_{\ell=1}^r g^\ell(t,X(t),X_t)dW^\ell(t),$$

with $(X(0),X_0)=(x,\xi)$ has a unique solution with $\sup_{s\le t}|X(s)|$ and $\sup_{s\le t}|X_s|\in L^p(\Omega,\mathcal{F}_t,P)$ for all $t\ge 0$ and $p<\infty$. We obtain in this section an integration by parts formula involving $X(t)$ and X_t under conditions sufficiently general. This formula is an extension of the formula which appears in Norris (10) as Theorem 2.3 without the delay variable X_t. This generality is needed for the iterations of the integration by parts formula involved in proving the smooth density result.

The integration by parts formula is obtained by viewing a perturbed solution of (2.9) in two ways. Observe that the map $v:[0,T]\times R^d\times L^2(J,\mathbb{R}^d)\to\mathbb{R}^r\times\mathbb{R}^m$ is C^∞ and bounded, with all its derivatives of polynomial growth. For $h\in\mathbb{R}^m$ and $\ell=1,2,\dots,r$ let $W^{h,\ell}(t)=W^\ell(t)+\int_0^t v^\ell(s,X(s),X_s)\dot{h}\,ds$ The perturbed process $X^h(t)$ is defined by

$$dX^h(t)=f\left(t,X^h(t),X_t^h\right)dt+\sum_{\ell=1}^r g^\ell\left(t,X^h(t),X_t^h\right)dW^{h,\ell}(t), \tag{2.15}$$

with $(X^h(0), X_0^h) = (x, \xi)$, or equivalently (writing $(v(s, X(s), X_s)\dot{h})^\ell$ for the ℓ^{th} component)

$$dX^h(t) = \left[f(t, X^h(t), X_t^h) + \sum_{\ell=1}^{r} g^\ell(t, X^h(t), X_t^h)(v(t, X(t), X_t)\dot{h})^\ell\right] dt + \sum_{\ell=1}^{r} g^\ell(t, X^h(t), X_t^h)\, dW^\ell(t). \quad (2.16)$$

Using Girsanov Theorem a new probability measure $\mathbb{P}^h$ may be found to make $W^h(t)$an$\mathbb{R}^r$ Brownian motion. Since $X(t)$ is a measurable function of the path $(W(s))_{s\le t}$, formula (2.15) implies that the law of $X^h(t)$ and X_t^h under $\mathbb{P}^h$ is independent of h; in other words

$\frac{\partial}{\partial h}\int_\Omega \Phi(t, X^h(t), X_t^h)\, d\mathbb{P}^h = 0$ for all $\Phi \in C_b([0,T] \times \mathbb{R}^d \times L^2(J, \mathbb{R}^d))$. Assume now v^ℓ and all its derivatives are bounded. Let $X^h(t)$ be the solution of (2.15)(216). Next, if we denote by $\mathbb{E}^h$ the expectation with respect to the measure $\mathbb{P}^h$, then, for all bounded $\mathcal{F}$-measurable random variables Y^h, we have

$$\mathbb{E}^h[Y^h] = \mathbb{E}[Z^h(t)Y^h] \quad (2.17)$$

where$Z^h(t)$ is given by

$$Z^h(t) = \exp\left[-\sum_{\ell=1}^{r}\int_0^t (v(s, X(s), X_s)\cdot h)^\ell\, dW^\ell(s) - \frac{1}{2}\int_0^t |v(s, X(s), X_s)\cdot h|^2\, ds\right] \quad (2.18)$$

and $\mathbb{P}^h = Z^h(t)\mathbb{P}$ on $\mathcal{F}_t$. Its derivative with respect h is given by :

$$\frac{\partial}{\partial h}Z^h(t) = -Z^h(t)(\int_0^t \sum_{\ell=1}^{r} v^\ell(s, X(s), X_s)dW^\ell(s). + \sum_{\ell=1}^{r}\int_0^t < v^\ell(s, X(s), X_s), h^\ell > v^\ell(s, X(s), X_s)ds). \quad (2.19)$$

Following the steps similar to those in (10) we make the following observations: If Y^h is a function of $(t, X^h(t), X_t^h)$, then by using equation (2.17) and the fact that the law of $X^h(t)$ and X_t^h under $\mathbb{P}^h$ is independent of h; we have

$$\frac{\partial}{\partial h}\mathbb{E}^h[Y^h] = \frac{\partial}{\partial h}\mathbb{E}[Z^h(t)Y^h] = 0, \quad (2.20)$$

which, after applying Leibnitz rule, implies

$$\mathbb{E}[\frac{\partial}{\partial h}Z^h(t)\cdot Y^h] + \mathbb{E}\left[Z^h(t)\frac{\partial}{\partial h}Y^h\right] = 0. \quad (2.21)$$

From (2.20) and(2.21) it follows that the expression $\mathbb{E}[Z^h(t)Y^h]$ does not depend on h and hence we have

$$\mathbb{E}[Z^h(t)Y^h] = \mathbb{E}[Z^0(t)Y^0] \quad (2.22)$$

Then by using equation (2.21)with $Y^h = \Phi(t, X^h(t), X_t^h)$ we find that

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$-\mathbb{E}[\frac{\partial}{\partial h}Z^h(t)\cdot Y^h]|_{h=0}$ Right side of equation (2.10)

And

$\mathbb{E}[Z^h(t)\frac{\partial}{\partial h}Y^h]|_{h=0}$ Left side of equation (2.10)

Thus we find that equation (2.10) holds.

Note that the process $V(t)$ (which appears in the formula for a possible choice of the function v^ℓ) is a $2d\times 2d$ matrix $V(t)=\begin{pmatrix}V_{11}(t) & V_{12}(t)\\ V_{21}(t) & V_{22}(t)\end{pmatrix}$ which solves the delay SDE ([E: V 3]). Similarly the process $U(t)$ is a given $2d\times 2d$ matrix $U(t)=\begin{pmatrix}U_{11}(t) & U_{12}(t)\\ U_{21}(t) & U_{22}(t)\end{pmatrix}$ which solves the delay SDE

$$d\begin{pmatrix}U_{11}(t) & U_{12}(t)\\ U_{21}(t) & U_{22}(t)\end{pmatrix}=\begin{pmatrix}h_x(t)U_{11}(t) & h_x(t)U_{12}(t)\\ h_x(t)U_{21}(t) & h_x(t)U_{22}(t)\end{pmatrix}+\begin{pmatrix}\int_J h_\xi(t,\vartheta)U_{11,t}(\vartheta)d\vartheta & \int_J h_\xi(t,\vartheta)U_{12,t}(\vartheta)\,d\vartheta\\ \int_J h_\xi(t,\vartheta)U_{21,t}(\vartheta)\,d\vartheta & \int_J h_\xi(t,\vartheta)U_{22,t}(\vartheta)\,d\vartheta\end{pmatrix}\tag{2.25}$$

We can also split (2.25) into the following four delay SDE's

$$dU_{11}(t)=h_x(t)U_{11}(t)+\int_J h_\xi(t,\vartheta)U_{11,t}(\vartheta)\,d\vartheta;\tag{2.26}$$

$$dU_{12}(t)=h_x(t)U_{12}(t)+\int_J h_\xi(t,\vartheta)U_{12,t}(\vartheta)\,d\vartheta;\tag{2.27}$$

$$dU_{21}(t)=h_x(t)U_{21}(t)+\int_J h_\xi(t,\vartheta)U_{21,t}(\vartheta)\,d\vartheta;\tag{2.28}$$

$$dU_{22}(t)=h_x(t)U_{22}(t)+\int_J h_\xi(t,\vartheta)U_{22,t}(\vartheta)\,d\vartheta.\tag{2.29}$$

Here $U_{ij}(t)$and $U_{ij,t}$, $i,\ j=1,2$ are the maps defined (with the same notations) as in the beginning of this chapter. For further applications or iterations of the Integration by Parts Formula, without loss of generality, we shall continue to use $X(t)$ and $\mathbb{P}$ instead of $X^h(t)$ and $\mathbb{P}^h$ respectively. Suppose that $U(t)V(t)=I$. Next we apply Itö formula (using the delay SDE for $U(t)$ and putting $(t)=-d<U(\cdot),V(\cdot)>(t)$) as follows

$$\begin{aligned}0 &= dU(t)V(t)+U(t)dV(t)+d<U(\cdot),V(\cdot)>(t)\\ &= h_x(t)+\int_J h_\xi(t,\vartheta)U_t(\vartheta)d\vartheta V(t)+U(t)dV(t)-A(t)\,dt\end{aligned}\tag{2.30}$$

Next we multiply equation (2.30)by $V(t)$ and use that $V(t)U(t)=I$ to get

$$V(t)h_x(t)+V(t)\int_J h_\xi(t,\vartheta)U_t(\vartheta)d\vartheta V(t)+dV(t)-V(t)A(t)\,dt=0\tag{2.31}$$

or, equivalently,

$$dV(t) \quad = V(t)A(t)\,dt - V(t)h_x(t) - V(t)\int_J h_\xi\,(t,\vartheta)U_t(\vartheta)d\vartheta V(t) \tag{2.32}$$

Then by looking at the martingale parts of the SDE for $U(t)$ namely

$$dU(t) = h_x(t)U(t) + \int_J h_\xi\,(t,\vartheta)U_t(\vartheta)\,d\vartheta \tag{2.33}$$

and the delay SDE in SDE (2.32) we can compute the covariance process in (2.30) and get

$$d < U(\cdot),V(\cdot) > (t)$$

$$\begin{aligned} &= -\sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)U(t)V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t - \sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)U(t)V(t)\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)\,d\vartheta V(t)d\,t \\ &\quad - \sum_{\ell=1}^{r}\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)\,d\vartheta V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t \\ &\quad - \sum_{\ell=1}^{r}\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)\,d\vartheta V(t)\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)\,d\vartheta V(t)d \; = -A(t), \end{aligned} \tag{2.34}$$

or, equivalently,

$$\begin{aligned} A(t) &= \sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t)\right)^2 d\,t + \sum_{\ell=1}^{r}\frac{\partial g^\ell}{\partial x}(t)\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)\,d\vartheta V(t)d\,t \\ &+ \sum_{\ell=1}^{r}\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)\,d\vartheta V(t)\frac{\partial g^\ell}{\partial x}(t)d\,t + \sum_{\ell=1}^{r}\left(\int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)\,d\vartheta V(t)\right)^2 d\,t \\ &= \sum_{\ell=1}^{r}\left(\frac{\partial g^\ell}{\partial x}(t) + \int_J\frac{\partial g^\ell}{\partial \xi}(t,\vartheta)U_t(\vartheta)\,d\vartheta V(t)\right)^2 dt. \end{aligned} \tag{2.35}$$

REMARKS

1. All the results which we have established in this work can be extended by replacing the Brownian motion W by another process $Z: [0,a] \times \Omega \to \mathbf{R}^{\mathbf{d}}$, $(d \in \mathbf{N})$ which is a continuous martingale adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$ and has independent increments and satisfies with some constant K the inequalities $|Z(t) - Z(s)\ \mathcal{F}_s| \leq K(t-s)$ and $\mathbf{E}\ (|Z(t) - Z(s)|^2\ \mathcal{F}_s \leq K(t-s)$ for $0 \leq s \leq t \leq a$. Observe that the above properties of Z which we have just mentioned are the only properties of W which we have used (in case of Brownian motion) to prove the results which we have obtained in this work. See , , and .
2. All the lemmas and theorems in this work hold for any delay interval $J' = [-r,0) \quad (r \geq 0)$ inplace of $J = [-1,0)$. See , , and .

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